International Journal of Theoretical Physics, Vol. 32, No. 6, 1993

Energy-Momentum Complex of Gravitational Field in the Palatini Formalism

Jan Novotný[1]

Received May 4, 1992

It is shown that Murphy's energy-momentum complex of the gravitational field, derived from the Hilbert Lagrangian by use of the Palatini formalism, is identical to the complex derived from the same Lagrangian in a standard way by Mitskievic. The explicitly tensorial formulation of conservation laws in general relativity is effectively used and some properties of the complex in question are discussed in connection with Murphy's article.

1. INTRODUCTION

Murphy recently proposed (Murphy, 1990) to define the energy-momentum complex of the gravitational field as

$$\mathfrak{K}_{\mu}^{\nu} = (\partial\mathfrak{L}_G/\partial\Gamma^{\alpha}_{\beta\sigma,\nu})\Gamma^{\alpha}_{\beta\sigma,\mu} - \delta^{\nu}_{\mu}\mathfrak{L}_G \tag{1}$$

where

$$\mathfrak{L}_G = \alpha \mathfrak{g}^{\mu\nu} R_{\mu\nu} = \alpha\sqrt{-g}\, R = \sqrt{-g}\, L_G \tag{2}$$

$\mathfrak{g}^{\mu\nu} = \sqrt{-g}\, g^{\mu\nu}$, $\alpha = \frac{1}{16}\pi$, and the Ricci tensor $R_{\mu\nu}$ is expressed with the help of Christoffel symbols as

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\alpha,\nu} - \Gamma^{\alpha}_{\mu\nu,\alpha} + \Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta} - \Gamma^{\beta}_{\mu\nu}\Gamma^{\alpha}_{\beta\alpha} \tag{3}$$

The expression (1) is the canonical energy-momentum complex of a gravitational field in the framework of the Palatini formalism, where the Hilbert invariant Lagrangian of the gravitational field (2) is expressed as $\mathfrak{L}_G(g, \Gamma, \partial\Gamma)$ instead of the standard $\mathfrak{L}_G(g, \partial g, \partial^2 g)$. As is well known, we obtain in this case not only the Einstein equations

$$\mathfrak{T}_{\mu}^{\nu} = \sqrt{-g}\, T_{\mu}^{\nu} = -2\alpha\sqrt{-g}(R_{\mu}^{\nu} - \tfrac{1}{2}R\delta_{\mu}^{\nu}) \tag{4}$$

[1]Department of Theoretical Physics and Astrophysics, Masaryk University, Kotlářská 2, 61137 Brno, Czechoslovakia.

but also the relations for the Levi-Civita connection

$$\Gamma^{\alpha}_{\beta\gamma} = \tfrac{1}{2} g^{\alpha\sigma}(g_{\sigma\beta,\gamma} + g_{\sigma\gamma,\beta} - g_{\beta\gamma,\sigma}) \tag{5}$$

as a consequence of the variational principle.

The fact that the use of the Palatini formalism does not change the physical content of the theory expressed by the relations (4), (5) represents a nontrivial property of the Hilbert Lagrangian. Now, the question arises whether this use of the Palatini formalism could lead to an essentially different formulation of the conservation laws. It is shown in Murphy's article that the expressions (1) differ from those obtained from the Einstein Lagrangian reduced on second derivatives of the metric. But it would be more natural to compare them with the expressions derived from the Hilbert Lagrangian (Mitskievic, 1961), namely

$$\mathfrak{K}^{\nu}_{\mu} = [(\partial \mathfrak{L}_G/\partial g_{\alpha\beta,\nu}) - (\partial \mathfrak{L}_G/\partial g_{\alpha\beta,\sigma\nu})_{,\sigma}] g_{\alpha\beta,\mu} + (\partial \mathfrak{L}_G/\partial g_{\alpha\beta,\sigma\nu}) g_{\alpha\beta,\sigma\mu} - \mathfrak{L}_G \delta^{\nu}_{\mu} \tag{6}$$

Our principal aim will be the demonstration of the identity of $\mathfrak{K}^{\nu}_{\mu}$ from (1) and from (6) after substitution of (5) into (1). Because the direct verification would be very tedious, we shall present two more accessible ways of this proof. Then we shall discuss some properties of the complex in question concerned in Murphy's paper, with regard to its identity with Mitskievic's complex.

2. PROOFS OF IDENTITY OF MURPHY AND MITSKIEVIC COMPLEXES

The canonical complex (6)—like the Einstein complex—can be derived from a superpotential as follows (Mitskievic, 1969):

$$\mathfrak{K}^{\nu}_{\mu} = \mathfrak{U}^{\nu\sigma}_{\mu,\sigma} - \mathfrak{T}^{\nu}_{\mu} \tag{7}$$

where $\mathfrak{T}^{\nu}_{\mu}$ is the energy-momentum complex of matter given by (4), and

$$\mathfrak{U}^{\nu\sigma}_{\mu} = -\mathfrak{U}^{\sigma\nu}_{\mu} = \alpha(\mathfrak{g}^{\nu\rho}\Gamma^{\sigma}_{\rho\mu} - \mathfrak{g}^{\sigma\rho}\Gamma^{\nu}_{\rho\mu}) \tag{8}$$

Considering (2) and (4), we conclude that the identity of $\mathfrak{K}^{\nu}_{\mu}$ from (1) and (7) can be derived under the supposition of the validity of the relation

$$\mathfrak{g}^{\alpha\beta}(\partial R_{\alpha\beta}/\partial \Gamma^{\rho}_{\kappa\lambda,\nu}) \Gamma^{\rho}_{\kappa\lambda,\mu} = \alpha^{-1} \mathfrak{U}^{\nu\sigma}_{\mu,\sigma} + 2\mathfrak{g}^{\nu\sigma} R_{\sigma\mu} \tag{9}$$

which can be verified by direct calculation using (3), (5), and (8). Nevertheless this calculation is still rather tedious. Therefore we give here another proof based on a manifestly covariant expression of the conservation laws which was developed in Novotný (1984, 1989).

Our starting point is the "first variational formula" for the Lagrangian (2) (Trautman, 1964)

$$\mathfrak{L}_G^{\alpha\beta}\bar{\delta}g_{\alpha\beta}-\delta\mathfrak{P}^{\nu}_{,\nu}=0 \tag{10}$$

where $\mathfrak{L}_G^{\alpha\beta}$ are Euler–Lagrange expressions corresponding to the Lagrangian (2) with respect to $g_{\alpha\beta}$ and

$$\delta\mathfrak{P}^{\nu}=-\mathfrak{L}_G\delta x^{\nu}-(\partial\mathfrak{L}_G/\partial\Gamma^{\alpha}_{\beta\gamma,\nu})\bar{\delta}\Gamma^{\alpha}_{\beta\gamma} \tag{11}$$

$$\bar{\delta}g_{\alpha\beta}=\delta g_{\alpha\beta}-g_{\alpha\beta,\sigma}\delta x^{\sigma} \tag{12}$$

$$\bar{\delta}\Gamma^{\alpha}_{\beta\gamma}=\delta\Gamma^{\alpha}_{\beta\gamma}-\Gamma^{\alpha}_{\beta\gamma,\sigma}\delta x^{\sigma} \tag{13}$$

Here $\delta g_{\alpha\beta}$ and $\delta\Gamma^{\alpha}_{\beta\gamma}$ are the variations of appropriate variables with respect to an infinitesimal transformation of the space-time manifold, which is given by the vector field δx^{α}. Let us emphasize that in (10) we already suppose the validity of relations (5) and consequently we omit the Euler–Lagrange expressions with respect to the $\Gamma^{\alpha}_{\beta\gamma}$.

A standard approach to the conservation laws begins by putting

$$\delta x^{\alpha}=\varepsilon\xi^{\alpha}(x^{\beta}) \tag{14}$$

where ε is a parameter, and ξ^{α} is the generator of a one-parameter group of transformations. We slightly modify this in order to obtain related equations in a manifestly covariant form. Therefore after expressing

$$\bar{\delta}g_{\alpha\beta}=-\varepsilon(\xi^{\sigma}_{,\beta}g_{\alpha\sigma}+\xi^{\sigma}_{,\alpha}g_{\sigma\beta}-g_{\alpha\beta,\sigma}\xi^{\sigma}) \tag{15}$$

we shall use the fact that the Lie variation $\bar{\delta}g_{\alpha\beta}$ is a tensor field and we shall write it with the help of a locally geodetic system as

$$\bar{\delta}g_{\alpha\beta}=-2\varepsilon\xi_{(\alpha;\beta)} \tag{16}$$

(parentheses mean symmetrization of indices). We apply a similar procedure also to the $\bar{\delta}\Gamma^{\alpha}_{\beta\gamma}$. Because in the expression

$$\bar{\delta}\Gamma^{\alpha}_{\beta\gamma}=\varepsilon(\Gamma^{\sigma}_{\beta\gamma}\xi^{\alpha}_{,\sigma}-\Gamma^{\alpha}_{\beta\sigma}\xi^{\sigma}_{,\gamma}-\Gamma^{\alpha}_{\sigma\gamma}\xi^{\sigma}_{,\beta}-\xi^{\alpha}_{,\beta\gamma}-\Gamma^{\alpha}_{\beta\gamma,\sigma}\xi^{\sigma}) \tag{17}$$

the first derivatives of $\Gamma^{\alpha}_{\beta\gamma}$ and the second derivatives of ξ^{α} are included, we must express (17) in the normal Riemannian coordinates (Misner *et al.*, 1973), where

$$\Gamma^{\alpha}_{\beta\gamma,\sigma}=-\tfrac{2}{3}R^{\alpha}_{(\beta\gamma)\sigma} \tag{18}$$

$$\xi^{\alpha}_{,\beta\gamma}=\xi^{\alpha}_{;\beta;\gamma}+\tfrac{2}{3}R^{\alpha}_{(\beta\sigma)\gamma}\xi^{\sigma} \tag{19}$$

Using also the well-known commutation relations, we obtain

$$\bar{\delta}\Gamma^{\alpha}_{\beta\gamma}=-\varepsilon[\xi^{\alpha}_{;(\beta;\gamma)}+\tfrac{2}{3}(R^{\alpha}_{(\beta\gamma)\sigma}-R^{\alpha}_{(\sigma\gamma)\beta})\xi^{\sigma}] \tag{20}$$

Remember that no term with first covariant derivatives of ξ^{α} is contained in (20).

Now we subtract (16) and (20) from (10) and divide by $\varepsilon\sqrt{-g}$ in order to obtain a vector field instead of a density. We also use [compare (4)]

$$\mathfrak{L}_G^{\alpha\beta} = \tfrac{1}{2} T^{\alpha\beta}\sqrt{-g} \tag{21}$$

and we express the resulting vector field with the help of a locally geodetic system. In this way we have

$$T^{\alpha\beta}\xi_{\alpha;\beta} + t^{\nu}_{;\nu} = 0 \tag{22}$$

where

$$\varepsilon\sqrt{-g}\, t^{\alpha} = \delta\mathfrak{P}^{\alpha} \tag{23}$$

In the first term of (22) we apply a step leading to the second theorem of Noether, i.e., we write

$$T^{\alpha\beta}\xi_{\alpha;\beta} = (T^{\alpha\beta}\xi_\alpha)_{;\beta} - T^{\alpha\beta}_{;\beta}\xi_\alpha \tag{24}$$

and as a consequence of the arbitrariness of ξ_α we obtain the strong conservation law

$$S^{\nu}_{;\nu} = 0 \tag{25}$$

where

$$S^{\nu} = T^{\nu\sigma}\xi_\sigma + t^{\nu} \tag{26}$$

The identical character of the strong law (25) is explicitly expressed by the introduction of the superpotential

$$S^{\nu} = U^{\nu\sigma}_{;\sigma} \tag{27}$$

where $U^{\alpha\beta}$ are components of an antisymmetric tensor field. These components can be found by a modification of the method used by Mitskievic (Mitskievic, 1969; see also Novotný, 1984, 1989). Let us write

$$S^{\nu} = B^{\alpha\nu}\xi_\alpha + B^{\alpha\beta\nu}\xi_{\alpha;\beta} + B^{\alpha\beta\gamma\nu}\xi_{\alpha;(\beta;\gamma)} \tag{28}$$

where $B^{\alpha\nu}$, $B^{\alpha\beta\nu}$, $B^{\alpha\beta\gamma\nu} = B^{\alpha\gamma\beta\nu}$ are tensor fields. Then

$$U^{\nu\sigma} = -U^{\sigma\nu} = (-B^{\alpha\nu\sigma} + \tfrac{2}{3}B^{\alpha\beta[\nu\sigma]}_{;\beta})\xi_\alpha - \tfrac{4}{3}B^{\alpha\beta[\nu\sigma]}\xi_{\alpha;\beta} \tag{29}$$

is a superpotential; square brackets mean antisymmetrization of indices. This can be easily proved as a consequence of identities following from (25) and from the independence of ξ_α, $\xi_{\alpha;\beta}$, $\xi_{\alpha;(\beta;\gamma)}$.

It remains to determine $B^{\alpha\nu\sigma}$, $B^{\alpha\beta[\nu\sigma]}$ in (28). It follows from (26), (23), (11), and (20) that

$$B^{\alpha\nu\sigma} = 0 \tag{30}$$

$$B^{\alpha\beta[\nu\sigma]} = (\partial\mathfrak{L}_G/\partial\Gamma^{\delta}_{\beta[\nu,\sigma]})g^{\alpha\delta} = \tfrac{3}{4}\alpha(g^{\alpha\nu}g^{\beta\sigma} - g^{\alpha\sigma}g^{\beta\nu}) \tag{31}$$

$$B^{\alpha\beta[\nu\sigma]}_{;\beta} = 0 \tag{32}$$

and consequently

$$U^{\nu\sigma} = -2\alpha\xi^{[\nu;\sigma]} \tag{33}$$

The superpotential (33) has a fundamental meaning for the conservation law theory. For example, Murphy's energy-momentum complex $\mathfrak{K}^{\nu}_{\mu}$ can be derived from it with the help of (7), where

$$\mathfrak{U}^{\nu\sigma}_{\mu} = \sqrt{-g}\, U^{\nu\sigma}(\xi^{\alpha}_{(\mu)}) \tag{34}$$

and

$$\xi^{\alpha}_{(\beta)} = \delta^{\alpha}_{\beta} \tag{35}$$

represents four generating fields of a natural basis in the given system of space-time coordinates.

Finally, we can perform analogous calculations for the Lagrange function (1) expressed as a function of metrics and its first and second derivatives. In this case we have instead of (11)

$$\begin{aligned} \delta\mathfrak{P}^{\nu} = -\mathfrak{L}_G\delta x^{\nu} + [(\partial\mathfrak{L}_G/\partial g_{\alpha\beta,\nu\sigma})_{,\sigma} - (\partial\mathfrak{L}_G/\partial g_{\alpha\beta,\nu})]\bar{\delta}g_{\alpha\beta} \\ -(\partial\mathfrak{L}_G/\partial g_{\alpha\beta,\nu\sigma})\bar{\delta}g_{\alpha\beta,\sigma} \end{aligned} \tag{36}$$

In a locally geodetic system, the middle term in (35) vanishes and

$$\bar{\delta}g_{\alpha\beta,\sigma} = -2\xi_{(\alpha;\beta);\sigma} \tag{37}$$

Using this fact and the commutation relations, we obtain an expression of the type (27) for S^{ν}, where

$$B^{\alpha\nu\sigma} = 0 \tag{38}$$

$$B^{\alpha\beta[\nu\sigma]} = \partial L_G/\partial g_{\alpha[\nu,\sigma]\beta} = \tfrac{3}{4}\alpha(g^{\alpha\nu}g^{\beta\sigma} - g^{\alpha\sigma}g^{\beta\nu}) \tag{39}$$

It is the same as (30), (31) and therefore we obtain also the same superpotential (33), (34) and the same energy-momentum complex $\mathfrak{K}^{\nu}_{\mu}$.

3. SOME CONSEQUENCES

According to the previous considerations, all conclusions made with respect to the Mitskievic complex are valid also for the complex proposed by Murphy. Let us remember that the best clue to its properties is provided by the superpotential (33), resp. (34). It is the well-known Komar superpotential (Komar, 1959) divided by the factor 2.

First let us calculate the density of the energy current in vacuum related to the generating field ξ^{σ}. It is

$$\mathfrak{K}^{\nu}_{0} = \sqrt{-g}\, U^{\nu\sigma}_{;\sigma} = -2\alpha\sqrt{-g}\, \xi^{[\nu;\sigma]}_{;\sigma} \tag{40}$$

where ξ^{σ} is the timelike field of the natural basis. It can be easily seen that (40) vanishes in the case of the Killing field, where

$$\xi^{(\nu;\sigma)} = 0 \tag{41}$$

That is, it vanishes in the static gravitational field in vacuum. But recall that it does not mean the zero value of the total mass-energy of a static insular system. This quantity can be calculated with the help of the superpotential as the zero component of the total four-momentum

$$P_0 = E = \int \mathfrak{N}_0^{\lambda}\, d\Sigma_{\lambda} = \frac{1}{2}\oint \mathfrak{U}_0^{\lambda\kappa}\, d\sigma_{\lambda\kappa} = \frac{1}{2}\oint \mathfrak{U}_0^{0i}\, d\sigma_i = \frac{M}{2} \tag{42}$$

(in the last integral the summation includes the space indices only). So the energy is equal to one-half of the Schwarzschild mass (putting $c = 1$). Therefore we cannot agree with Murphy's statement that "the mass of an isolated particle vanishes." In fact, this mass is given by (42). Not even the energy flux from an isolated system diverges. As shown by Novotný and Horský (1983), the "quadruple formula" can be derived also for the Mitskievic complex, only the loss of energy by radiation is half in comparison with the loss calculated with the help of the Einstein or Landau-Lifshitz complex. But because also the total energy (41) is half, this difference has no physically observable effect and it can be simply treated as a different definition of energy.

Let us recall that the Mitskievic and consequently also the Murphy formulation suffer essential difficulty: the four-momenta calculated on the hypersurfaces of relative simultaneity connected by the Lorentz transformation are generally not the same, but

$$P'_{\alpha} - P_{\alpha} = -v_k \oint \mathfrak{U}_{\alpha}^{kl}\, d\sigma_l \tag{43}$$

holds, where v_k is (three-dimensional) velocity of an asymptotically inertial system Σ' with respect to another such system Σ. In other words, P_{α} does not represent a vector with respect to the Lorentz transformation if $\oint \mathfrak{U}_{\alpha}^{kl}\, d\sigma_l \neq 0$. But this is the case for the Mitskievic–Murphy complex. See also the discussion on the differential and integral conservation laws in general relativity (Møller, 1961; Kovacs, 1985; Novotný, 1987), suggesting that, unfortunately, success in the localization of gravitational conserved quantities is paid for by defects in their integral behavior.

So we can conclude that the use of the Palatini formalism in the theory of conservation laws is remarkable from the point of view of its brevity and simplicity, but it does not lead to an essentially new complex of energy-momentum. Therefore it probably would not throw any new light on old problems.

ACKNOWLEDGMENTS

The essential part of this work was performed during the author's stay at the Department of Theoretical Physics, University in Bern. The author wishes to express his thanks for the invitation and the kind hospitality he received. He is especially indebted to Prof. Petr Hájíček. He is also grateful to Dr. M. Lenc, Brno, for his critical remarks.

REFERENCES

Komar, A. (1959). *Physical Review*, **113**, 934.

Kovacs, D. (1985). *General Relativity and Gravitation*, **17**, 927.

Misner, C. W., Thorne, K. S., and Wheeler, J. A. (1973). *Gravitation*, Freeman, San Francisco, Chapter 11.

Mitskievic, N. V. (1961). *Annals of Physics*, **12**, 118.

Mitskievic, N. V. (1969). *Physical Fields in the General Theory of Relativity*, Moscow [in Russian].

Møller, C. (1961). *Annals of Physics*, **12**, 118.

Murphy, G. (1990). *International Journal of Theoretical Physics*, **29**, 1003.

Novotný, J. (1984). *Coll. Math. Soc. J. Bolyai* (Debrecen), **46**, 959.

Novotný, J. (1987). *General Relativity and Gravitation*, **10**, 1043.

Novotný, J. (1989). In *Conference on Differential Geometry and its Applications*, J. Janyška and D. Krupka, eds., World Scientific, Singapore, p. 383.

Novotný, J., and Horský, J. (1983). In *Proceedings of the Einstein Foundation Int.*, Vol. 2, Nagpur, India.

Trautman, A., Pirani, F. A. E., and Bondi, H. (1964). Lectures on General Relativity, Brandeis Summer Institute in Theoretical Physics, New Jersey.